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On Generalising Abraham-Lorentz Equation†

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Abstract

It is proved that runaway solutions persist if Abraham's force $-m(\ddot{\mathbf{x}} - \tau\dddot{\mathbf{x}})$ is generalised by adding to it a *finite* number of terms which are linear in higher derivatives of $\ddot{\mathbf{x}}$. The implication of this result to Eliezer's relativistic generalisation of the Lorentz-Dirac equation is discussed.

The non-relativistic Abraham-Lorentz equation of motion for a radiating charged particle

$$\mathbf{F}_{\mathrm{ext}}(t) = m\ddot{\mathbf{x}} - \frac{2e^2}{3c^3}\dddot{\mathbf{x}} \equiv m(\ddot{\mathbf{x}} - \tau\dddot{\mathbf{x}}) \tag{1}$$

has runaway solutions or preaccelerated solutions (see, e.g., Erber, 1961, Jackson, 1962, Panofsky *et al.*, 1962, Rohrlich, 1965). It seems natural to add terms involving higher derivatives of $\mathbf{x}(t)$ and see whether these runaway solutions disappear. After trying this unsuccessfully with a few terms, I arrived at the following theorem:

Theorem: an equation of motion of the form

$$\mathbf{F}_{\mathrm{ext}}(t) = m\left[\mathbf{a} - \tau\dot{\mathbf{a}} + \ldots + c_n \frac{d^N\mathbf{a}}{dt^N}\right]$$

$$\equiv m\sum_{n=0}^{N} c_n \frac{d^n\mathbf{a}}{dt^n} \tag{2}$$

will always have runaway solutions, where $\tau > 0$ and $c_n(n \geqslant 2)$ are arbitrary real constants.

This theorem tells us that if we seek a generalisation of equation (1) which does not have runaway (or preaccelerated) solutions, we should either add an

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infinite number of linear terms in the velocity and its derivatives or add non-linear terms in the velocity and its derivatives.

An example of a linear generalisation of equation (1) which does *not* have runaway solutions is provided by the quasi-stationary approximation (Erber, 1961, Herglotz, 1903, Wildermuth, 1955) for the extended Lorentz election:

$$F_{\text{ext}}(t) = \frac{e^2}{3r_0^2c}\,[x(t) - \dot{x}(t - 2r_0/c)]$$

$$= \frac{2}{3}\frac{e^2}{r_0c^2}\ddot{x} - \frac{2}{3}\frac{e^2}{c^3}\dddot{x} - \frac{e^2}{3r_0^2c}\sum_{n=3}^{\infty}(-2r_0/c)^n/n!\,\frac{d^n\dot{x}(t)}{dt^n} \tag{3}$$

Our theorem tells us that breaking this series after a finite number of terms will introduce runaway solutions. It thereby illustrates the fact that perturbation expansions to any order could lead to formally unphysical results even though the exact expression is formally physical.†

Non-linear generalisations of (1) are given by the relativistic Lorentz-Dirac equation (Dirac, 1938).

$$(F_{\text{ext}})_\alpha(s) = mu_\alpha^{(1)} - \frac{2}{3}\frac{e^2}{c^3}(u_\alpha^{(2)} + u^{(1)^2}u_\alpha) \tag{4}$$

and its generalisation by Eliezer (1947).

$$(F_{\text{ext}})_\alpha(s) = mu_\alpha^{(1)} - \frac{2}{3}\frac{e^2}{c^3}(u_\alpha^{(2)} + u^{(1)^2}u_\alpha)$$

$$+ \sum_{n=1}^{\infty} B_{2n}[u_\alpha^{(2n+1)} - (u, u^{(2n+1)})u_\alpha - \{(u, u^{(2n)}) \tag{5}$$

$$+ (-1)^n(u^{(n-1)}, u^{(n-1)}) + \tfrac{1}{2}(-1)^n u^{(n)^2}\}u_\alpha^{(1)}]$$

$$+ \cdots.$$

where, B_{2n} are arbitrary constants, $(F_{\text{ext}})_\alpha$ the external Minkowski force, $u_\alpha^{(n)}$ the nth derivatives of the four-velocity u_α with respect to the proper time s, and $(a, b) \equiv a_0b_0 - \mathbf{a}\,.\,\mathbf{b}$. Equation (4), whose non-relativistic limit (where only linear terms in $\mathbf{v}$ and its derivatives are kept) is given by equation (1), has runaway solutions.‡ However, it is not known in general whether Eliezer's equation (5) with *finite* number of terms ($B_{2n} = 0$ for $n > N$) will also have runaway solutions. By the above theorem, it will certainly have

† There are also examples where just the opposite happens: sometimes perturbation expansions are more physical than the exact solutions. For example, some wave equations for spins $s = \frac{3}{2}$ with electromagnetic interactions have accausal exact solutions, but their perturbation expansions are causal to all orders (Velo, 1972, Velo *et al.*, 1971); the accausality here means $v > c$ and is different from ours, which is associated with pre-acceleration. I would like to thank H. Biritz for the above reference.

‡ Runaway solutions of the relativistic equations correspond to the Minkowski velocity $\mathbf{u}$ reaching infinity, and thus the usual velocity $\mathbf{v} = (1 + (u/c)^2)^{-1/2}\mathbf{u}$ approaching the speed of light as $t \to \infty$.

runaway solutions in the non-relativistic limit. It is not clear, however, whether this fact implies the existence of runaway solutions in the relativistic case as well.

Proof of the theorem: Since equation (2) is linear, it will have runaway or preaccelerated solutions under the action of external force, if it has a runaway solution for the homogeneous case. Hence, it is enough to prove that

$$0 = \sum_{n=0}^{N} c_n \frac{d^n \mathbf{a}}{dt^n} \tag{6}$$

has a runaway solution. Substituting

$$\mathbf{a}(t) = \mathbf{A}\, e^{-i\omega t} \tag{7}$$

into equation (6) gives

$$0 = 1 - \tau(-i\omega) + \ldots + c_N(-i\omega)^N \equiv f(\omega) \tag{8}$$

Factorising the polynomial $f(\omega)$ gives

$$\begin{aligned} f(\omega) &= c_N(-i)^N \prod_{n=1}^{N} (\omega - \omega_n) \\ &= c_N(-i)^N \prod_{n=1}^{N} (-\omega_n) \cdot \left[1 - \left(\sum^{N} \frac{1}{\omega_n} \right) \omega + \ldots \right] \end{aligned} \tag{9}$$

Comparing the linear term in ω in equations (8) and (9) gives

$$\tau = i \sum \frac{1}{\omega_n} = i \sum \frac{\omega_n^*}{|\omega_n|^2} = \sum \frac{\text{Im}\, \omega_n}{|\omega_n|^2} \tag{10}$$

where the last step follows from $f(-\omega^*) = f^*(\omega)$, which tells us that the zeros of $f(\omega)$ are either pure imaginary $\omega = i\,\text{Im}\,\omega$, or occur in pairs $\omega_\pm = \pm\,\text{Re}\,\omega + i\,\text{Im}\,\omega$. Since $\tau > 0$, the sum $\Sigma\, \text{Im}\, \omega_n / |\omega_n|^2$ cannot be positive, unless at least one $\text{Im}\, \omega_n$ is positive. Such an ω_n leads to a runaway solution when substituted into (7).

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